FULL GROUPS AND SOFICITY

GÁBOR ELEK

Abstract. First, we answer a question of Pestov, by proving that the full group of a sofic equivalence relation is a sofic group. Then, we give a short proof of the theorem of Grigorchuk and Medynets that the topological full group of a minimal Cantor homeomorphism is LEF. Finally, we show that for certain non-amenable groups all the generalized lamplighter groups are sofic.

1. Introduction

1.1. **Sofic groups and LEF groups.** The notion of sofic groups was introduced by Weiss [12] and Gromov [5] (in a somewhat different form) . A group Γ is sofic if for any finite set $F \subset \Gamma$ and $\epsilon > 0$ there exists a finite set A and a mapping $\Theta : \Gamma \to Map(A)$ such that ([3])

- If $f, g, fg \in F$ then $d_H(\Theta(fg) - \Theta(f)\Theta(g)) \leq \epsilon$, where
$$d_H(\alpha, \beta) = \frac{|\{x \in A \mid \alpha(x) \neq \beta(x)\}|}{|A|}.$$
- If $1 \neq f \in F$ then $d_H(\Theta(f), 1) > 1 - \epsilon$.
- $\Theta(1) = 1$.

All amenable and residually finite groups are sofic. It is an open question whether non-sofic groups exist. If we add the extra requirement that $\Theta(fg) = \Theta(f)\Theta(g)$, then we get the class of LEF-groups (locally embeddable into finite groups). This class of groups was introduced by Gordon and Vershik [11]. Clearly, all residually finite groups are LEF. However, simple, finitely presented groups are not LEF. Nevertheless, by a recent result of Juschenko and Monod [6] (and Theorem 2), there exist simple, finitely generated LEF-groups.

1.2. **Sofic equivalence relations.** Let $X = \{0, 1\}^{\mathbf{N}}$ be the standard Borel space with the natural product measure μ. Let $\Phi : \mathbf{F}_\infty \curvearrowright X$ be a (not necessarily free) Borel action of the free group of countably infinite generators $\{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1} \dots\}$ preserving μ. Note that $\mathbf{F}_\infty = \cup_{r=1}^{\infty} \mathbf{F}_r$, where $\mathbf{F}_r$ is the free group of rank r. Hence, we also have probability measure preserving (p.m.p) Borel actions $\Phi_r : \mathbf{F}_r \curvearrowright X$. We say that $x, y \in X$ are equivalent, $x \sim_\Phi y$ if there exists $w \in \mathbf{F}_\infty$, such that $w(x) = y$. Note that slightly abusing the notation we write $w(x)$ instead of $\Phi(w)(x)$. Thus, the action Φ represents a countable measured equivalence relation E_Φ on X. Similarly, each Φ_r represents a countable measured

Work supported in part by a Marie Curie grant, TAMOP 4.2.1/B-09/1/KMR-2010-003 and MTA Renyi "Lendulet" Groups and Graphs Research Group.

AMS SUBJECT CLASSIFICATION:20F65, 37A20

equivalence relation E_{Φ_r} on X, and $E_\Phi = \cup_{i=1}^\infty E_{\Phi_r}$. Each equivalence relation E_{Φ_r} defines a graphing [7] G_r on X:

- $V(G_r) = X$.
- $(x,y) \in E(G_r)$ if $\gamma_i x = y$ or $\gamma_i y = x$ for some i (so, there may be loops in G_r).

Observe that each component of G_r is a countable graph of bounded vertex degrees. We label each directed edge (x,y) with all the generators mapping x to y. Thus an edge, even a loop, may have multiple labels.
Now let us consider transitive actions of $\mathbf{F}_r$ on countable sets. If $\alpha : \mathbf{F}_r \curvearrowright Y$ is such an action then we have a bounded degree graph structure on Y with multiple labels on the edges from the set $\{\gamma_1, \gamma_1^{-1}, \dots, \gamma_r, \gamma_r^{-1}\}$. Let T_r be the set of graphs of all countable $\mathbf{F}_r$-actions with a distinguished vertex (the root) such that all the vertices are labeled by the elements of $\{0,1\}^r$. Let $G \in T_r$. We define the the k-ball around the root x, $B_k(x)$ as the induced subgraph on vertices of G in the form of $w(x)$, where $w \in \mathbf{F}_r$ is a reduced word of length at most k. That is, $B_k(x)$ is the ball centered at x of radius k with respect to the shortest path metric of G. The ball $B_k(x)$ is a finite rooted graph with edge-colors from the set $\{\gamma_1, \gamma_1^{-1}, \dots, \gamma_r, \gamma_r^{-1}\}$ and vertex labels from the set $\{0,1\}^r$. We denote the set of all possible k-balls arising from $\mathbf{F}_r$-actions by U_r^k. We can define a compact metric structure on the set T_r the following way. Let $d_r(G,H) = \frac{1}{2^k}$ if k is the maximal number such that the k-balls around the roots of G resp. H are isomorphic as rooted, labeled graphs.

Observe that if $\Theta : \mathbf{F}_\infty \curvearrowright X$ is a p.m.p action then for each $r \geq 1$ and $x \in X$ one can associate an element $G(\Theta, x) \in T_r$. Namely, the orbit graph of x, where the vertex labels are given by the X-values, restricted on the first r coordinates. Thus, we have a Borel map $\pi_\Theta : X \to T_r$. For $\kappa \in U_r^k$, let $\mu^k_{\Theta_r}(\kappa) = (\pi_\theta)_\star(\mu)(L_\kappa)$, where $L_\kappa \subset T_r$ is the set of elements G such that the k-ball around the root of G is isomorphic to κ. In other words, $\mu^k_{\Theta_r}(\kappa)$ is the probability that the k-ball around a μ-random element of X is isomorphic to κ. Now let $\alpha : \mathbf{F}_r \curvearrowright Y$ be an $\mathbf{F}_r$-action on a finite set. Then for each element y of Y, we can associate an element of T_r. Namely, Y itself with root y. Hence, we can define a probability distribution $\mu^{k,r}_\alpha$ on U_r^k. Following [1] we say that the action $\Theta : \mathbf{F}_\infty \curvearrowright X$ is sofic if for all $r \geq 1$, there exists a sequence of finite $\mathbf{F}_r$-actions $\{\alpha_n\}_{n=1}^\infty$ such that for each $k \geq 1$ and $\kappa \in U_r^k$

$$\lim_{n\to\infty} \mu^{k,r}_{\alpha_n}(\kappa) = \mu^k_{\Theta_r}(\kappa)\,.$$

In [1] the authors proved that

- Soficity is a property of the underlying equivalence relations. That is, if an action Θ_1 is orbit equivalent to a sofic action Θ_2, then Θ_2 is sofic as well.
- Treeable equivalence relations are sofic.
- Actions associated to Bernoulli shifts of sofic groups are sofic.

1.3. **Full groups.** Let $E(X,\mu)$ be a countable, measured equivalence relation on a Borel set X with invariant measure μ. The Borel full group of E is the group $[E]_B$ of all Borel bijections $T : X \to X$ such that for any $x \in X$, $T(x) \sim_E x$. We call two such bijections

T_1, T_2 equivalent if

$$\mu(\{x \in X \mid T_1(x) = T_2(x)\}) = 1\,.$$

The measurable full group $[E]$ is the group formed by the equivalence classes. Obviously, $[E] = [E]_B/N$, where N is the normal subgroup of elements in $[E]_B$ fixing almost all points of X.
Now, let $T : C \to C$ be a homeomorphism of the Cantor set C. The topological full group $[[T]]$ is the group of homeomorphisms $S : C \to C$ such that C can be partitioned into finitely many clopen sets $C = \cup_{i=1}^n A_i$ such that $S_{|A_i} = T^{n_i}$ for some integer n_i.

1.4. **Results.** Answering a question of Pestov [1], we prove the following theorem.

Theorem 1. *The measurable full group of a sofic equivalence relation is sofic.*

Then, we give a very short proof of a result of Grigorchuk and Medynets [4].

Theorem 2. *The topological full group of a minimal Cantor homeomorphism is LEF.*

Let X be a countably infinite set and Γ be a countable group acting faithfully and transitively on X. Then Γ can be represented by automorphisms on the Abelian group $\oplus_{x\in X}\{0,1\}$. The groups $\oplus_{x\in X}\{0,1\} \rtimes \Gamma$ are called the lamplighter group of the Γ-action. If the action is the natural translation action on Γ, then we get the classical lamplighter group of Γ. Paunescu [10] proved that if Γ is sofic, then the classical lamplighter group $\oplus_{\gamma\in\Gamma}\{0,1\} \rtimes \Gamma$ is sofic. If Γ is amenable, then all its generalized lamplighter groups are amenable hence sofic. Nevertheless, we show that there exist non-amenable groups for which all the generalized lamplighter groups are sofic.

Theorem 3. *Let Γ^k be the k-fold free product of the cyclic group of two elements. Then, for any transitive, faithful action of Γ^k on a countable set the associated lamplighter group is LEF.*

Acknowledgement: The author thanks Nicolas Monod and Gábor Pete for valuable discussions.

2. Compressed sofic representations

Let Γ be a countable sofic group with elements $\{\gamma_1, \gamma_2, \dots\}$. A compressed sofic representation of Γ is defined the following way. For any $i \geq 1$, we have a constant $\epsilon_i > 0$ and for any $n \geq 1$ we have mappings $\Theta_n : \Gamma \to Map(A_n)$ such that $|A_n| < \infty$ satisfying the following condition: For all $r > 0$ and $\epsilon > 0$ there exists $K_{r,\epsilon} > 0$ such that if $n > K_{r,\epsilon}$ then

- $d_H(\Theta_n(\gamma_i\gamma_j)\Theta_n(\gamma_i)\Theta_n(\gamma_j)) < \epsilon$ if $1 \leq i, j \leq r$.
- $d_H(\Theta_n(\gamma_i), Id) > \epsilon_i$ if $1 \leq i \leq r$.

Thus, in a compressed sofic representation we allow large amount of fixed points for each $\gamma \in \Gamma$.

Lemma 2.1. *If Γ has a compressed sofic representation then Γ is sofic.*

[1] MR2566316-MathSciNet Review

Proof. Let $\tilde{\Theta}^k_n : \Gamma \to Map(A^k_n)$ be defined by

$$\tilde{\Theta}^k_n(\gamma)(x_1, x_2, \dots, x_k) = (\Theta_n(\gamma)(x_1), \Theta_n(\gamma)(x_2), \dots)\,.$$

Observe that if $\gamma, \delta \in \Gamma$, then

- $d_H(\tilde{\Theta}^k_n(\gamma\delta), \tilde{\Theta}^k_n(\gamma)\tilde{\Theta}^k_n(\delta)) \leq (1 - d_H(\Theta_n(\gamma\delta), \Theta_n(\gamma)\Theta_n(\delta))^k$
- $d_H(\tilde{\Theta}^k_n(\gamma), Id) > 1 - (1 - d_H(\Theta_n(\gamma), Id))^k$

Hence, we can choose ϵ, n and k appropriately to obtain for any $F \subset \Gamma$ and $\epsilon' > 0$ a map Θ as in the Introduction, proving the soficity of Γ. $\square$

3. The proof of Theorem 1

Let $\Phi : \mathbf{F}_\infty \curvearrowright \{0,1\}^{\mathbf{N}}$ be a sofic action preserving the product measure μ. Let $\Gamma \subset [E]$ be a finitely generated group, where $[E]$ is the equivalence relation defined by Φ. So, we have an action $\Phi_\Gamma : \Gamma \curvearrowright \{0,1\}^{\mathbf{N}}$. Our goal is to construct a compressed sofic representation of Γ. Let $\{\gamma_n\}_{n=1}^\infty$ be an enumeration of the elements of Γ. Let $\epsilon_n = \mu(Fix(\Phi_\Gamma(\gamma_n))/2$. Since Γ is in the full group, $\epsilon_n > 0$. Now, fix a subset $F \subseteq \Gamma$ and $\epsilon > 0$. We need to construct a map $\Theta : F \to Map(A)$ for some finite set A such that if $\gamma_i, \gamma_j, \gamma_i\gamma_j \in F$ then

$$d_H(\Theta(\gamma_i\gamma_j)\Theta(\gamma_i)\Theta(\gamma_j)) < \epsilon \tag{1}$$

$$d_H(\Theta(\gamma_i), 1) > \epsilon_i \tag{2}$$

Let $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}\}$ be a symmetric generating set for Γ. Observe that we have an action $\Sigma_\Gamma : \mathbf{F}_m \curvearrowright \{0,1\}^{\mathbf{N}}$ preserving μ such that $\Sigma_\Gamma(\delta) = \Phi_\Gamma(\tau(\delta))$, where $\tau : \mathbf{F}_m \to \Gamma$ is the natural quotient map. A dyadic E-map of depth k is a Borel map $Q : X \to X$ is defined the following way. For each $\rho \in \{0,1\}^k$ we pick $w_Q(\rho) \in \mathbf{F}_k \subset \mathbf{F}_\infty$ and define $Q(x) = \Phi(w_Q(\rho))(x)$ if the first k-coordinate of x is ρ.
A dyadic approximation of Γ is a sequence of families $\{Q_k(s_i)\}_{i=1}^m, \{Q_k(s_i^{-1})\}_{i=1}^m$, where for any $1 \leq i \leq m$

- $Q_k(s_i) : X \to X$, $Q_n(s_i^{-1}) : X \to X$ are dyadic E-maps of depth k.
- $\lim_{k\to\infty} \mu(\{x \in X \mid Q_k(s_i)(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0$
- $\lim_{k\to\infty} \mu(\{x \in X \mid Q_k(s_i^{-1})(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0$

We do not require Q_k to be a bijection. Nevertheless, Q_k can be extended to a homomorphism from $\mathbf{F}_m$ to $Map(X)$. Note that for simplicity we identified the generating set of $\mathbf{F}_m$ by the set $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}\}$.

Since all the $\Sigma_\Gamma(s_i)'s$ are Borel bijections such dyadic approximations clearly exist. The following lemma is an immediate consequence of the definition of the dyadic approximation.

Lemma 3.1. *For any $\delta \in \mathbf{F}_m$*

$$\lim_{k\to\infty} \mu(Fix(Q_k(\delta))) = \mu(Fix(\Sigma_\Gamma(\delta)))\,.$$

Proposition 3.1. *There exists a sequence of mappings $\hat{\Theta}_k : \mathbf{F}_m \to Map(B_k)$, where $|B_k| < \infty$ such that for any $\delta \in \mathbf{F}_m$*

$$\lim_{k\to\infty}(\mu(Fix(Q_k(\delta))) - \frac{|Fix(\hat{\Theta}_k(\delta))|}{|B_k|}) = 0\,.$$

That is

$$\lim_{k\to\infty}\frac{|Fix(\hat{\Theta}_k(\delta))|}{|B_k|} = \mu(Fix(\Sigma_\Gamma(\delta)))\,.$$

Proof. Let $\Phi_k : \mathbf{F}_k \curvearrowright \{0,1\}^{\mathbf{N}}$ be the restriction of Φ. Since Φ is sofic, there exists a sequence of mappings $\{\iota^n_k : \mathbf{F}_k \curvearrowright Perm(C_{k,n})\}^\infty_{n=1}$, where $C_{k,n}$ is a finite $\{0,1\}^k$-vertex labeled graph such that for any $t \geq 1$ and $\kappa \in U^t_k$

$$\lim_{n\to\infty}\mu^{t,k}_{\iota^n_k}(\kappa) = \mu^t_{\Phi_k}(\kappa)\,.$$

Recall that Q_k is not necessarily an action, only a homomorphism from $\mathbf{F}_m$ to $Map(X)$. Hence, the local statistics of Q_k can not be described using the elements of U^t_k as in the case of honest $\mathbf{F}_m$-actions. So, let W^t_k be the set of isomorphism classes of rooted t-balls of vertex degrees at most $2m$, where the vertices are labeled by elements of the set $\{0,1\}^k$ and the edges (possibly loops) are labeled by subsets of $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_m, s_m^{-1}\}$. Note that $U^t_k \subset W^t_k$. Let $x, y \in X$ be points such that $B^{\Phi_k}_{k^2}(x)$ and $B^{\Phi_k}_{k^2}(y)$ represent the same element in $U^{k^2}_k$. Here $B^{\Phi_k}_{k^2}(x)$ denotes the k-ball with respect to the graphing associated to Φ_k. Then, by the definition of the dyadic approximations $B^{Q_k}_k(x)$ and $B^{Q_k}_k(y)$ represent the same elements in W^k_k. Now we construct a sequence of maps $\hat{\Theta}^n_k : \mathbf{F}_m \curvearrowright Map(C_{k,n})$ the following way.

$$\hat{\Theta}^n_k(s_i)(x) = \iota^n_k(w_{Q_k(s_i)}(\rho(x)))(x)\,,$$

where $\rho(x)$ is the $\{0,1\}^k$-label of x. By the previous observation, for any $\delta \in \mathbf{F}_m$

$$\lim_{n\to\infty}\frac{|Fix(\hat{\Theta}^n_k(\delta))|}{|C_{k,n}|} = \mu(Fix(Q_k(\delta)))\,.$$

This finishes the proof of the proposition $\square$

Pick a section $\sigma : \Gamma \to \mathbf{F}_m$, that is a map such that $\tau\sigma = Id$. Let $\hat{\Theta}_k$ as in Proposition 3.1. Define $\Theta_k : \Gamma \to Map(B_k)$ by

$$\Theta_k(\gamma) = \hat{\Theta}_k(\sigma(\gamma))\,.$$

Then $\{\Theta_k\}^\infty_{k=1}$ is a compressed sofic representation of Γ. $\square$

4. The proof of Theorem 2

Let $T : C \to C$ be a minimal homeomorphism and $\Gamma \subset [[T]]$ be a finitely generated subgroup of the topological full group of T with symmetric generating set $S = \{a_1, a_2, \dots, a_k\}$. It is enough to prove that Γ is LEF. Let $x \in C$ and consider the T-orbit $\{T^n(x)\}^\infty_{-\infty}$. We define the map $\phi : \Gamma \to Perm(\mathbf{Z})$ of Γ into the permutation group of the integers the following way. Let $\phi(\gamma)(n) = m$, if $\gamma(T^n(x)) = T^m(x)$. Since T acts freely on C, ϕ is well-defined.

Lemma 4.1. *ϕ is an injective homomorphism.*

Proof. If $\phi(\gamma) = Id$, then γ fixes all the elements of the orbit of x. Since all the orbits are dense, this implies that $\gamma = 1$. The fact that ϕ is a homomorphism follows immediately, since ϕ is the restriction of the Γ-action onto the orbit of x. $\square$

Let $a = \max |n|$, where for some $p \in C$ and $a_i \in S$, $a_i(p) = T^n(p)$. We define a sequence

$$l : \mathbf{Z} \to \{-a, -a+1, \dots, 0, 1, \dots, a-1, a\}^S$$

the following way. Let $l(n) := (t_{a_1}, t_{a_2}, \dots, t_{a_k})$, where $a_i(T^n(x)) = T^{n+t_{a_i}}(x)$. The following lemma is well-known, we prove it for the sake of completeness.

Lemma 4.2. *l is a repetitive sequence, that is, if we find a substring σ in l, then there exists $m \geq 1$ such that for any interval of length m we can find σ.*

Proof. For a point $p \in C$, we can define its n-pattern

$$q_n(p) := \{-n, -n+1, \dots, 0, 1, \dots, n-1, n\} \to \{-a, -a+1, \dots, a-1, a\}$$

by $q_n(p)(j) := (t_{a_1}, t_{a_2}, \dots, t_{a_k})$, where $a_i(T^j(x)) = T^{j+t_{a_i}}(x)$. Observe that the set of points with a given n pattern is closed. Now, let us suppose that for a sequence $\{k_r\}_{r=1}^\infty \subset \mathbf{Z}$ the intervals $(k_r - r, k_r + r)$ do not contain σ as a substring. Then, if z is a limit point of $\{T^{k_r}(x)\}_{r=1}^\infty$, no translates of z have σ as a part of their n-patterns. Therefore the orbit closure of z does not contain x, in contradiction with the minimality of T. $\square$

Now let $r \geq 1$ and consider the string $\sigma_r = l_{|\{-ar, -ar+1, \dots, ar-1, ar\}}$, where a is the constant defined above. Note that if $\gamma \in \Gamma$ is the product of at most r generators then $|\phi(\gamma)(i) - i| \leq ar$. Pick $n > 10a^r$ such that

- $l_{|\{-ar+n, -ar+1+n, \dots, ar-1+n, ar+n\}} = \sigma_r$,
- for any $\gamma \in \Gamma$ that is the product of at most r generators there is $0 < j < n$ such that $\gamma(j) \neq j$.

Now we define $\phi_r : W^r \to Perm(\mathbf{Z}_n)$, where W^r is the set of elements in Γ that are products of at most r generators by $\phi_r(i) = \phi(i)(mod\, n)$. Clearly, ϕ_r is injective and if $x, y, xy \in W^r$ then $\phi_r(x)\phi_r(y) = \phi_r(xy)$. This implies that Γ is LEF. $\square$

5. The proof of Theorem 3

Let $\alpha : \Gamma^k \to X$ be a transitive and faithful action of the free product group. Consider the Schreier graph G_α of the action with respect to the generators of the k cyclic groups $\{a_1, a_2, \dots a_k\}$. Recall that $V(G_\alpha)$ is X and $(x, y) \in E(G)$ if $y = a_i x$ for some $i \geq 1$. Hence G_α is a connected graph of vertex degree bound k.

Proposition 5.1. *Let α be as above. Then for any $1 \neq w \in \Gamma^k$, there exist infinitely many $y \in X$ such that $\alpha(w)(y) \neq y$.*

Proof. We will need the following lemma.

Lemma 5.1. *For any finite set $S \subseteq X$, there exists $g \in \Gamma^k$ such that $gS \cap S = \emptyset$.*

Proof. We define a lazy random walk on X the following way. For $y \in X$ the transition probability $p(x,y) = l/k$, where l is the number of generators a_i such that $a_i x = y$. It is well-known (see e.g. [9],[8]) that the probabilities $p_n(x,y)$ tend to zero for each pair $x, y \in X$. Now consider the standard random walk on the Cayley graph of Γ^k, the k-regular tree. Let $P_n(g)$ be the probability being at g after taking n steps starting from the identity. Then,

$$p_n(x,y) = \sum_{g \in \Gamma, gx=y} P_n(g)\,.$$

By the previous observation, if n is large enough, then

$$\sum P_n(g) < 1\,,$$

where the summation is taken for all $g \in \Gamma^k$ such that $gx \in S$, for some $x \in S$. Hence, there exists $g \in \Gamma^k$ such that $gS \cap S = \emptyset$. □

Now let us suppose that $w \in \Gamma^k$ fixes all points of X outside a finite set S. That is $\alpha(w)(S) = S$. Let $gS \cap S = \emptyset$. Then gwg^{-1} fixes all the points of X outside gS. Therefore the commutator $[w, gwg^{-1}]$ fixes all elements of X, in contradiction with the assumption that the action is faithful. □

Now fix a vertex $x \in X$ and consider the ball of radius n, $B_n(x)$ around x. We define an action $\alpha_n : \Gamma^k \curvearrowright B_n(x)$ the following way. Let $\partial B_n(x)$ be the boundary of the ball $B_n(x)$, that is, the set of all $y \in B_n(x)$ such that there exists a_i for which $\alpha(a_i)y \notin B_n(x)$. If $y \notin \partial B_n(x)$, then let $\alpha_n(a_i)y = \alpha(a_i)y$. If $y \in \partial B_n(x)$ and $\alpha(a_i)y \notin B_n(x)$, then let $\alpha_n(a_i)(y) = y$. Finally, if $y \in \partial B_n(x)$ and $\alpha(a_i)y \in B_n(x)$, then let $\alpha_n(a_i)(y) = \alpha(a_i)(y)$. Now let $L^n_k = \{0,1\}^{B_n(x)} \rtimes_{\alpha_n} \alpha_n(\Gamma^k)$ be the associated finite lamplighter group and $L^k = \oplus_{x \in X}\{0,1\} \rtimes_\alpha \Gamma^k$. Our goal is to embed L^k into L^n_k locally. That is, for any finite set $F \subset L^k$ we construct an injective map $\Theta : F \to L^n_k$ such that $\Theta(fg) = \Theta(f)\Theta(g)$. Recall, that each element of L^k can be uniquely written in the form $a \cdot w$, where $a \in \oplus_{x\in X}\{0,1\}$ and $w \in \Gamma^k$. We regard the elements of the lamplighter group as permutations of the set $\oplus_{x\in X}\{0,1\}$. If $\kappa \in \oplus_{x \in X}\{0,1\}$ and $p \in X$ then

$$(a \cdot w)(\kappa)_{|p} = a(p) + \kappa(\alpha(w^{-1})(p))\,.$$

We will also use the product formula

$$(a_2 \cdot w_2)(a_1 \cdot w_1) = (a_2 + \alpha(w_2)(a_1), w_2 w_1)\,,$$

where $\alpha(w_2)(a_1)(q) = a_1(\alpha(w_2^{-1})(q))$. For $l \geq 1$, let H_l be the set of elements of L^k in the form of $a \cdot w$, where w is a word of length at most l and the support of a is contained in $B_l(x)$. For $n \geq l$ we define the map $\tau^n_l : H_l \to L^n_k$ by $\tau^n_l(a \cdot w) := a \cdot \alpha_n(w)$.

Lemma 5.2. *If n is large enough then τ^n_l is injective.*

Proof. If n is large enough then $B_n(x)$ contains a point y such that

- $\alpha(w)(y) \neq y$
- $d(y, \partial B_n(x)) > l$
- $d(y, B_l(x)) > l$,

where d is the shortest path distance on the Schreier graph G_α. Let $\kappa \in \oplus_{x\in X}\{0,1\}$ be the element which is 1 at y and zero otherwise. Then

$$\tau_l^n(a \cdot w)(\kappa)_{|\alpha_n(w)(y)} = 1\,,$$

hence $\tau_l^n(a \cdot w)$ is not trivial. $\square$

The following lemma finishes the proof of Theorem 3.

Lemma 5.3. *Suppose that* $(a_1 \cdot w_1), (a_2 \cdot w_2)$ *and* $(a_2 \cdot w_2)(a_1 \cdot w_1) \in H_l$ *and* n *is large enough. Then*

$$\tau_l^n((a_2 \cdot w_2))\tau_l^n((a_1 \cdot w_1)) = \tau_l^n((a_2 \cdot w_2)(a_1 \cdot w_1))\,.$$

Proof. We need to prove that

$$(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1)) = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2 w_1)$$

holds in L_k^n. Fix an element $\kappa \in \{0,1\}^{B_n(x)}$. Let $n > 10l$ and $d(p, \partial B_n(x))) > 5l$. Then

$$(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)_{|p} = (a_2 \cdot w_2)(a_1 \cdot w_1)(\overline{\kappa})_{|p}$$

and

$$(a_2 + \alpha(w_2)(a_1) \cdot \alpha_n(w_2 w_1))(\kappa)_{|p} = (a_2 + \alpha(w_2)(a_1) \cdot (w_2 w_1)(\overline{\kappa})_{|p}\,,$$

where $\overline{\kappa}$ is an extension of κ onto X. On the other hand, if $d(p, \partial B_n(x))) \leq 5l$, then

$$(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)_{|p} = \alpha_n(w_2)\alpha_n(w_1)(\kappa)_{|p} =$$
$$= \alpha_n(w_2 w_1)(\kappa)_{|p} = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2 w_1))(\kappa)_{|p} \qquad \square$$

References

[1] G. Elek and G. Lippner, Sofic equivalence relations. *Journal of Functional Analysis* **258** (2010) no. 5, 1692–1708.

[2] G. Elek and E. Szabó, Hyperlinearity, essentially free actions and L^2-invariants. The sofic property. *Math. Ann.* **332** (2005), no. 2, 421–441

[3] G. Elek and E. Szabó, On sofic groups. *Journal of Group Theory* **9** (2006), no.2, 161–171

[4] R. Grigorchuk and K. Medynets, On algebraic properties of topological full groups. (preprint) http://arxiv.org/pdf/1105.0719.pdf

[5] M. Gromov, Endomorphisms of symbolic algebraic varieties. *J. Eur. Math. Soc* **1** (1999), 109–197.

[6] K. Juschenko and N. Monod, Cantor systems, piecewise translations and simple amenable groups. (preprint) http://arxiv.org/pdf/1204.2132.pdf

[7] A. S. Kechris and B. D. Miller, Topics in orbit equivalence. Lecture Notes in Mathematics **1852** Springer-Verlag (2004)

[8] R. Lyons, Asymptotic enumeration of spanning trees. *Combin. Probab. and Comput.* **14** (2005) no. 4, 491–522.

[9] B. Morris and Y. Peres, Evolving sets, mixing, and heat kernel bounds. *Probab. Theory Related Fields* **133** (2005) no. 2, 245–266.

[10] L. Paunescu, On sofic actions and equivalence relations. *Journal of Functional Analysis*, **261** (2011) no. 9, 2461–2485.

[11] A. M. Vershik and E. I. Gordon, Groups that are locally embeddable in the class of finite groups. *Algebra i Analiz* **9** (1997) no. 1, 71–97.
[12] B. Weiss, Sofic groups and dynamical systems. *Sankhya Ser. A* **62** (2000), 350–359.

gabor.elek@epfl.ch